MATH 2850: 'REVIEW' OF PARTIAL DERIVATIVE MECHANICS

RECALL: The **derivative** of a function f, f' is $\dfrac{df}{dx} = f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, provided this limit exists.

If $f'(a)$ exists, then, geometrically, $f'(a)$ is the slope of the tangent line at $(a, f(a))$.

More generally, $f'(a)$ is the instantaneous rate of change of f with respect to x when $x = a$.

DEFINITION: Given a function $f(x, y)$:

- the **partial** derivative of f **with respect to** x is $\dfrac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$
- the **partial** derivative of f **with respect to** y is $\dfrac{\partial f}{\partial y} = f_y(x, y) = \lim_{k \to 0} \dfrac{f(x, y+k) - f(x, y)}{k}$

provided these limits exist.

COMPUTATIONALLY:

To find $\dfrac{\partial f}{\partial x}$, take the derivative of f treating x as a variable and y as a constant.

To find $\dfrac{\partial f}{\partial y}$, take the derivative of f treating y as a variable and x as a constant.

EXAMPLE: Let $f(x, y) = 3x^2 y - \cos(xy)$. Find and simplify:

- $\dfrac{\partial f}{\partial x}$

Ans: $\dfrac{\partial f}{\partial x} = 6xy + y\sin(xy)$

- $f_y(x, y)$

Ans: $f_y(x, y) = 3x^2 + x\sin(xy)$

EXAMPLE: The Ideal Gas Law states that $PV = nRT$. In this equation, P denotes the pressure of the gas, V is the volume of the gas, T is the temperature of the gas, and n is the number moles (which gives the number of) gas molecules. The quantity R here is a constant.

- Solve the Ideal Gas Law for P and find $\dfrac{\partial P}{\partial V}$.

Ans: $P = \dfrac{nRT}{V}$ so $\dfrac{\partial P}{\partial V} = -\dfrac{nRT}{V^2}$

- Solve the Ideal Gas Law for V and find $\dfrac{\partial V}{\partial T}$.

Ans: $V = \dfrac{nRT}{P}$ so $\dfrac{\partial V}{\partial T} = \dfrac{nR}{P}$

- Solve the Ideal Gas Law for T and find $\dfrac{\partial T}{\partial P}$.

Ans: $T = \dfrac{PV}{nR}$ so $\dfrac{\partial T}{\partial P} = \dfrac{V}{nR}$

- Show $\left(\dfrac{\partial P}{\partial V}\right)\left(\dfrac{\partial V}{\partial T}\right)\left(\dfrac{\partial T}{\partial P}\right) = -1$.

IMPLICIT DIFFERENTIATION USING PARTIAL DERIVATIVES:

THEOREM: If $z = f(x, y)$ is a differentiable function of x and y and the equation $f(x, y) = c$ implicitly describes y as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}$$

EXAMPLE: Find an expression for $\frac{dy}{dx}$ in in terms of x and y if $x^2 + 2xy + y^3 = 9$.

Ans: $\frac{dy}{dx} = -\frac{2x + 2y}{2x + 3y^2}$

EXAMPLE: Find $\frac{dy}{dx}$ for the family of curves: $x^2 - y^2 = c^2$. What restrictions are needed?

Ans: $\frac{dy}{dx} = \frac{x}{y}$ provided $y \neq 0$.

DIFFERENTIAL EQUATIONS TERMINOLOGY:

DEFINITIONS:

A **differential equation** (DE) is an equation involving a function and one or more of its derivatives.

$$\frac{dy}{dx} = x\sin(y) \qquad y'' + 3y' + 2y = 0 \qquad \frac{\partial U}{\partial t} = 16\,\frac{\partial^2 U}{\partial x^2}$$

An **ordinary** differential equation (ODE) involves a function of a **single** variable.

A **partial** differential equation (PDE) involves a function of **several** variables.

ODEs: $\dfrac{dy}{dx} = x\sin(y)$, $y'' + 3y' + 2y = 0$ **PDE**: $\dfrac{\partial U}{\partial t} = 16\,\dfrac{\partial^2 U}{\partial x^2}$

The **order** of a DE is the highest derivative in the DE.

First Order DE: $\dfrac{dy}{dx} = x\sin(y)$ **Second Order DEs:** $y'' + 3y' + 2y = 0$, $\dfrac{\partial U}{\partial t} = 16\,\dfrac{\partial^2 U}{\partial x^2}$

NOTE: Every integration problem is a first order ODE! Finding $\displaystyle\int f(x)\,dx$ is equivalent to solving $y' = f(x)$.

A **solution** to an ODE is a **function** which satisfies the ODE on some **open** interval, (a, b).

The largest open interval for which a function is a solution is called the **interval of validity** of the solution.

A **solution curve** of an ODE is the graph of a solution of an ODE.

An **integral curve** is the graph of a curve (not necessarily a function) the restrictions of which are solution curves.

NOTE: Every solution curve is an integral curve but not vice-versa.

EXAMPLE: Show $x^2 - y^2 = 1$ is an integral curve for the ODE: $y' = \dfrac{x}{y}$, provided $y \neq 0$.

Find at least four solution curves and the associated intervals of validity.

Ans: Solution curves: $y = \sqrt{x^2 - 1}$ and $y = -\sqrt{x^2 - 1}$ each on one of the two intervals; $(1, \infty)$ or $(-\infty, -1)$

EXAMPLE: Find solutions to $y' = x\,e^{x/2}$ and the associated intervals of validity.

Ans: $y = \int x\,e^{x/2}\,dx = 2xe^{x/2} - 4e^{x/2} + C$, $(-\infty, \infty)$.

NOTES:

- Since each choice of 'C' yields a solution to the DE, we say $y = 2xe^{x/2} - 4e^{x/2} + C$ is a **<u>one</u> parameter family** of solutions to the DE. The word 'parameter' here refers to the choice of the constant C.
- Thanks to the Mean Value Theorem, we know **every** function which satisfies $y' = x\,e^{x/2}$ must be of the form $y = 2xe^{x/2} - 4e^{x/2} + C$. Hence we say $y = 2xe^{x/2} - 4e^{x/2} + C$ is the **general solution** of the ODE.

Our goal, whenever possible, is to describe the general solution to an ODE. 'Gotta Catch 'Em All!"

EXAMPLE: Consider the ODE: $\dfrac{dy}{dx} = y\,(3 - y)$.

- Verify $y = \dfrac{3C}{C + e^{-3x}}$ is a one parameter family of solutions to this ODE.

- Verify $y = 3$ is a solution to this ODE.

- Is $y = \dfrac{3C}{C + e^{-3x}}$ the general solution to the ODE? Why or why not?

INITIAL AND BOUNDARY VALUE PROBLEMS

EXAMPLE: Solve $y' = x\, e^{x/2}$. subject to $y(0) = 1$.

In a previous example, we found $y = 2xe^{x/2} - 4e^{x/2} + C$ is the general solution to $y' = x\, e^{x/2}$.

If we impose the condition $y(0) = 1$, then $-4 + C = 1$ so $C = 5$. Hence, our solution is: $y = 2xe^{x/2} - 4e^{x/2} + 5$.

The condition $y(0) = 1$ is called an **initial condition** or IC. A DE paired with an IC is called an **initial value problem** or IVP. More generally, an IVP is a DE along with solution information about **one** specific x-value.

EXAMPLE: Consider $y'' + 4y = 0$.

- Verify $y = c_1 \sin(2x) + c_2 \cos(2x)$ is a two parameter family of solutions to this ODE.

- Find a solution in the family satisfying $y'(\pi) = 0$ and $y(\pi) = 2$.

- What happens when you look for solutions in the family satisfying $y(0) = 0$ and $y(\pi) = 2$?

- What happens when you look for solutions in the family satisfying $y(0) = 0$ and $y(\pi) = 0$?

NOTE: In the first case, we have an IVP since we're given information about one particular x-value, $x = \pi$. The second and third cases, we are given information about two different x-values, $x = \pi$ and $x = 0$. In this case, we have a **boundary value problem** or BVP. As we'll see, IVPs tend to be 'nicer' in that we are typically guaranteed **unique** (one and only one) solutions. BVPs can have none, one, or infinitely solutions.

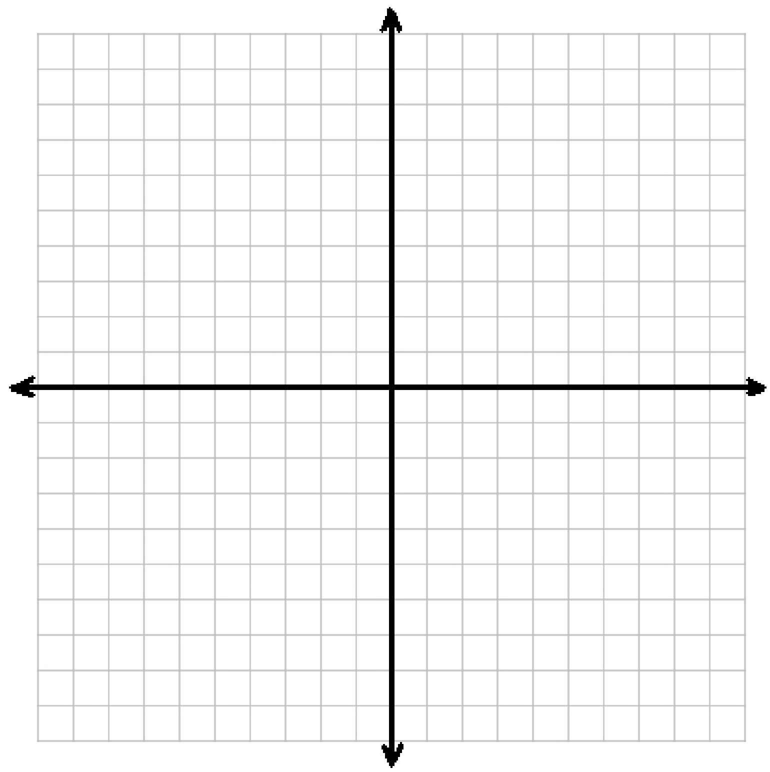
VISUALIZING FIRST ORDER DIFFERENTIAL EQUATIONS: SLOPE (DIRECTION) FIELDS

Suppose we are given a DE of the form: $\dfrac{dy}{dx} = f(x,y)$.

Geometrically we are looking for curves whose slope at (x,y) is given by $f(x,y)$. Hence, we may sample $f(x,y)$ at various points (x,y) in the plane and sketch a small '**lineal element**' which represents a portion of the tangent line to the solution curve at that point. The resulting graph is called a **slope field** or **direction field** for the DE.

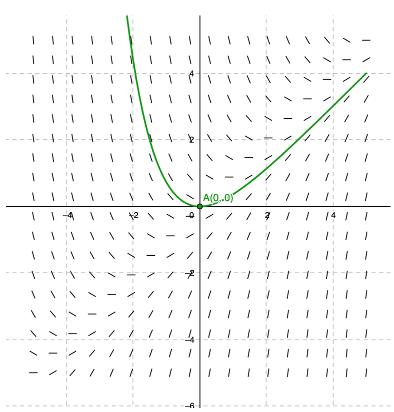
EXAMPLE: Sketch a direction field for $\dfrac{dy}{dx} = x - y$ on the graph below.

(x,y)	$f(x,y) = x - y$

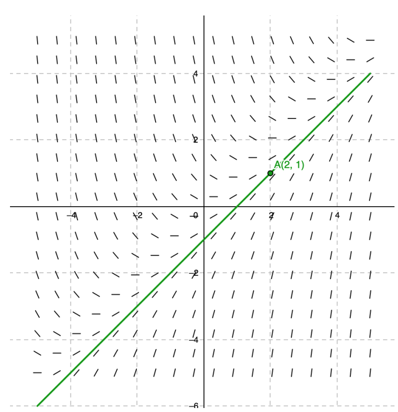


Use your direction field to sketch some solution curves corresponding to the ICs: $y(0) = 0$, $y(2) = 1$, $y(2) = -1$

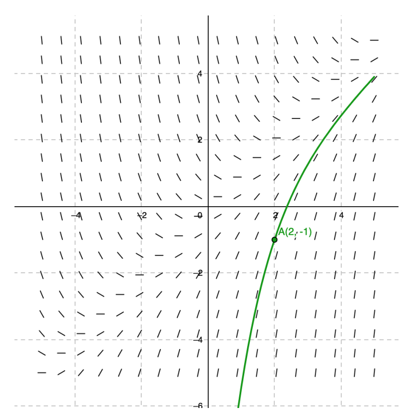
Ans:



$y(0) = 0$ solution



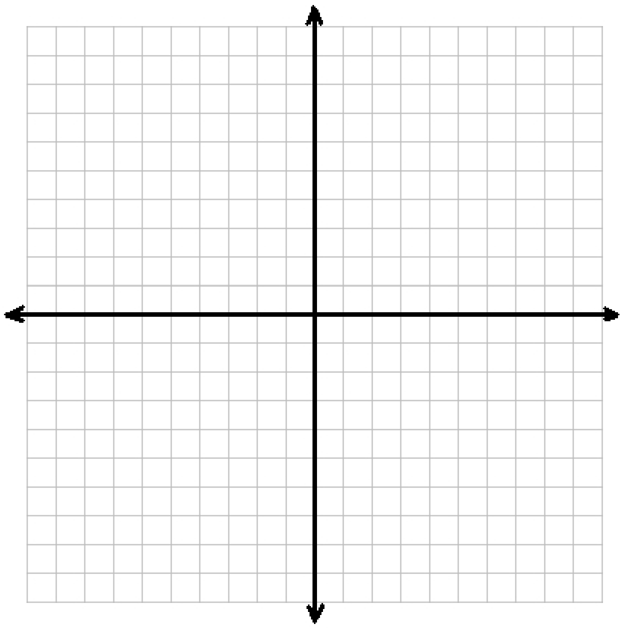
$y(2) = 1$ solution



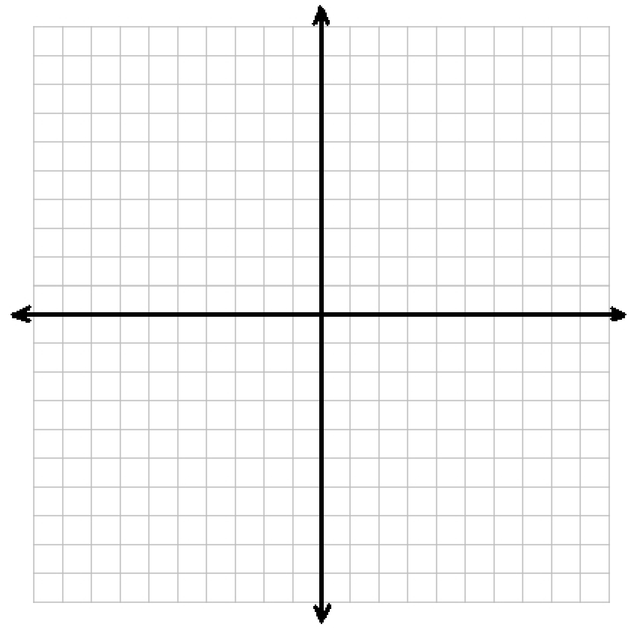
$y(2) = -1$ solution

EXAMPLE: Use a graphing utility to sketch direction fields for the given DE. Sketch solutions to the given IVP.

- $\dfrac{dy}{dx} = y\,(3-y)$; $y(0)=1$ and $y(0)=3$.

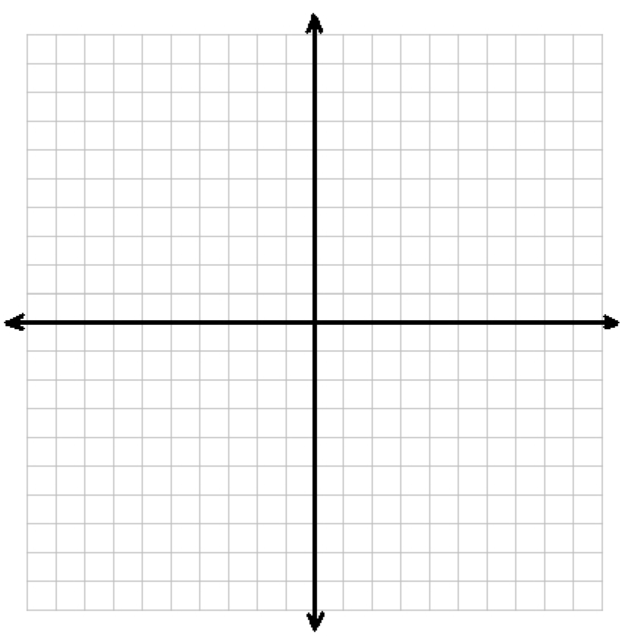


$y(0)=1$

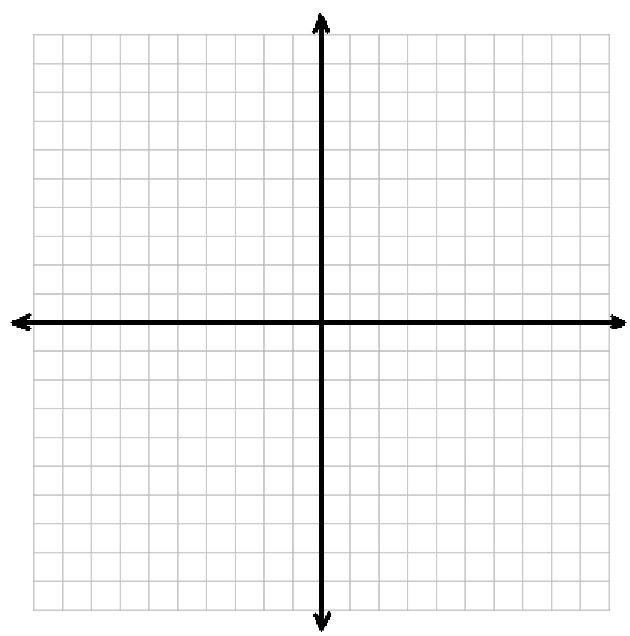


$y(0)=3$

- $\dfrac{dy}{dx} = \dfrac{x}{y}$; $y(0)=1$ and $y(2)=1$.



$y(0)=1$



$y(2)=1$

NOTE: Show $x^2 - y^2 = 3$ satisfies the DE and the condition $y(2)=1$. What gives?

HINT: $\dfrac{dy}{dx} = \dfrac{x}{y}$ is undefined when $y=0$. This leads to some 'interesting' scenarios - especially at $(0,0)$.

HOMEWORK: pg. 14: 1-8 all; pg. 26: 1-21 every other odd.